Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 144 (2004) 133–145
Topology and its Applications
www.elsevier.com/locate/topol

Countable compactness of hyperspaces and Ginsburg's questions

Jiling Cao [a,*,1], Tsugunori Nogura [b], A.H. Tomita [c,2]

[a] *Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland, New Zealand*
[b] *Department of Mathematical Sciences, Faculty of Science, Ehime University, Matsuyama 790-8577, Japan*
[c] *Departamento de Matemática, Institute of Mathematics and Statistics, University of São Paulo, 05315 São Paulo, Brazil*

Abstract

In this paper, we study the countable compactness and pseudocompactness of the hyperspace 2^X of a Hausdorff space X consisting of all nonempty closed subsets of X equipped with the Vietoris topology. Some open questions posed by Ginsburg in 1975 are considered. In particular, we give partial solutions to one of them.

MSC: primary 54D20; secondary 03E35, 54B20

Keywords: Countably compact; Homogeneous; Pseudocompact; Vietoris topology

1. Introduction

All topological spaces are assumed to be Hausdorff. Given a space X, 2^X denotes the collection of all nonempty closed subsets of X. One of the most important and well-studied topologies on 2^X is the Vietoris topology τ_V, which is also known as the finite topology. To describe this topology, we need some notation. For a subset E of X, let $E^- = \{A \in 2^X\colon\ A \cap E \neq \emptyset\}$, and $E^+ = \{A \in 2^X\colon\ A \subseteq E\}$. Then τ_V has as a subbase

* Corresponding author.
E-mail addresses: cao@math.auckland.ac.nz (J. Cao), nogura@ehimegw.dpc.ehime-u.ac.jp (T. Nogura), tomita@ime.usp.br (A.H. Tomita).
[1] Research supported by a post-doctoral fellowship from Japan Society for the Promotion of Science nominated by New Zealand government between 2000 and 2002.
[2] Research supported by a scholarship from the Ministry of Education of Japan in 2001.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2004.05.001

all subsets of 2^X of the forms U^- and V^+, where U and V are open subsets of X. For any finite family $\mathcal{V}$ of subsets of X, let

$$\langle\mathcal{V}\rangle = \Big\{F \in 2^X\colon\ F \subseteq \bigcup \mathcal{V},\ \ F \cap V \neq \emptyset \text{ for any } V \in \mathcal{V}\Big\}.$$

It is a well-known fact that the collection of all subsets of 2^X of the form $\langle\mathcal{V}\rangle$, where $\mathcal{V}$ is a finite family of open subsets of X, is a base for τ_V. From now on, the hyperspace 2^X of X will always carry the topology τ_V except it is stated explicitely.

One of the fundamental problems in the theory of hyperspaces is to decide how a topological property of X can be transfered to 2^X and vice versa. For instance, the famous Vietoris–Michael theorem [11, Theorem 4.2] asserts that a space X is compact if and only if 2^X is compact (see [2,9–11] for more results of this type). So, it is quite natural to ask the following question: *What can we say about the hyperspace 2^X of a countably compact or pseudocompact space X*? *Are there any analogs to the Vietoris–Michael theorem for countable compactness or pseudocompactness*? Recall that a space X is said to be *countably compact* if every infinite subset has an accumulation point; and X is said to be *pseudocompact* if every continuous real-valued function on it is bounded. When it is Tychonoff, a space is pseudocompact if and only if every sequence of nonempty open subsets has an accumulation point. It seems that unlike those covering properties considered in [10], the behavior of countable compactness-like properties with respect to the Vietoris topology is not easy to handle. In 1975, Ginsburg [6] considered the above question and discovered that the countable compactness (pseudocompactness) of 2^X has some nice connections to the countable compactness (pseudocompactness) of powers of X. Note that neither countable compactness nor pseudocompactness is (finitely) multiplicative in the realm of Tychonoff spaces as Novák [12] and Terasaka [14] showed independently. What Ginsburg proved are in fact the following:

(i) *If all powers of a space X are countably compact, then its hyperspace 2^X is countably compact*;
(ii) *If the hyperspace 2^X of a space X is countably compact, then all finite powers of X are countably compact*.

A result similar to (ii) also holds for pseudocompact Tychonoff spaces. In addition to these mentioned results, Ginsburg also showed that there is a Tychonoff space X all of whose finite powers are countably compact but whose hyperspace 2^X is not pseudocompact. Indeed, X is one of spaces constructed by Frolik in [5] with the following properties: All finite powers of X are countably compact, but X^ω is not pseudocompact. In this paper, we shall provide a Tychonoff space (see Example 2.4) all of whose finite powers are countably compact and whose countable infinite power is pseudocompact, but whose countable infinite power is not countably compact. The main purpose of this paper is to tackle the following question.

Question 1.1 [6, Remark 3.2]. *Is there any relation between the pseudocompactness* (*countable compactness*) *of X^ω and that of 2^X*?

In attempting to attack Question 1.1, we obtain some partial solutions to it. More precisely, we shall give a counterexample to this question in one direction in Section 2, and then show some positive results in the other direction in Section 3. Furthermore, we also show that under **MA**, $2^{\mathfrak{c}}$ is the best possible cardinal for powers of a countably compact space X to guarantee the countable compactness of 2^X. Several remarks related to hyperspaces of countably compact (pseudocompact) spaces are given in the last section of the paper.

2. Some examples

In this section, we shall extend the constructions of [6, Example 3.1] and [15, Example 4.13] to a more general approach which not only can be used to produce examples of countably compact spaces whose hyperspaces with τ_V are not countably compact, but also can be applied to prove positive results for some special classes of topological spaces.

Given any family $\{X_i\colon i \in A\}$ of countably compact spaces, let $\bigoplus\{X_i\colon i \in A\}$ be the disjoint union of X_i's. We define a countably compact space X as follows:

(i) If $|A| < \omega$, just let $X = \bigoplus\{X_i\colon i \in A\}$;
(ii) If $|A| \geqslant \omega$, we pick up an arbitrary point $\infty \notin \bigoplus\{X_i\colon i \in A\}$, and then endow $X = \bigoplus\{X_i\colon i \in A\} \cup \{\infty\}$ with a topology such that each X_i with its original topology is clopen in X, and such that every neighbourhood of ∞ contains all but finitely many of the X_i. In this case, the space X is usually called the *one-point countably compactification* of $\bigoplus\{X_i\colon i \in A\}$. It can be checked easily that X is regular (Tychonoff) if and only if all X_i's are regular (Tychonoff).

Theorem 2.1. *Let $\{X_i\colon i \in A\}$ be a family of countably compact spaces, and let X be the space defined above. If 2^X is countably compact, then the product space $\prod\{X_i\colon i \in A\}$ is countably compact.*

To prove Theorem 2.1, we need the following general lemma which shall be used in Section 3 as well.

Lemma 2.2. *Let X be a space, and let $\alpha > 0$ be a cardinal. If there are two α-sequences $(U_\xi\colon \xi < \alpha)$ and $(V_\xi\colon \xi < \alpha)$ of nonempty open sets in X; and a closed discrete sequence $(x_n\colon n < \omega)$ of points in the αth power X^α of X, where $x_n = \langle x_n(\xi)\rangle_{\xi<\alpha}$ for each $n < \omega$, such that*

(i) *$(U_\xi\colon \xi < \alpha)$ is pairwise disjoint,*
(ii) *$\overline{V}_\xi \subseteq U_\xi$ for all $\xi < \alpha$, and*
(iii) *$x_n(\xi) \in V_\xi$ for all $\xi < \alpha$ and all $n < \omega$, then 2^X is not countably compact.*

Proof. For each $n < \omega$, let $\bar{x}_n$ denote the closure of $\{x_n(\xi)\colon \xi < \alpha\}$ in X. We shall show that $(\bar{x}_n\colon n < \omega)$ has no accumulation points in 2^X, and thus 2^X is not countably compact.

Suppose that it has an accumulation point $F \in 2^X$. We first establish that $|F \cap U_\xi| = 1$ for every $\xi < \alpha$. If this is not true, then there are two possible cases for us to consider.

Case 1. There exists some $\xi' < \alpha$ such that $F \cap U_{\xi'} = \emptyset$. In this case, $F \in (X \setminus \overline{V}_{\xi'})^+$. Thus, $\bar{x}_n \in (X \setminus \overline{V}_{\xi'})^+$ for infinitely many $n < \omega$. This is impossible, as $x_n(\xi') \in V_{\xi'}$ for all $n < \omega$.

Case 2. There exists some $\xi'' < \alpha$ such that $|F \cap U_{\xi''}| > 1$. In this case, F must meet two disjoint open sets $G_0, G_1 \subseteq U_{\xi''}$. It follows that $(G_0)^- \cap (G_1)^-$ is a τ_V-open neighborhood of F, and thus $\bar{x}_n \in (G_0)^- \cap (G_1)^-$ for infinitely many $n < \omega$. This implies that for infinitely many $n < \omega$, we have both $\{x_n(\xi)\colon \xi < \alpha\} \cap G_0 \neq \emptyset$ and $\{x_n(\xi)\colon \xi < \alpha\} \cap G_1 \neq \emptyset$ simultaneously. Again, it is impossible, since every term x_n has exactly one coordinate contained in $U_{\xi''}$.

Now, let $F \cap U_\xi = \{x(\xi)\}$ for each $\xi < \alpha$. Then, we define a point $x \in X^\alpha$ such that $x = \langle x(\xi) \rangle_{\xi < \alpha}$. Next, we take an arbitrary basic open neighborhood

$$O = \prod_{i=0}^{k} O_{\xi_i} \times \prod \bigl\{ X_\xi\colon \xi \in \alpha \setminus \{\xi_0, \xi_1, \ldots, \xi_k\} \bigr\}$$

of the point x in X^α, where $k < \min\{\alpha, \omega\}$, each O_{ξ_i} $(i \leqslant k)$ is an open neighborhood of $x(\xi_i)$ in X and $X_\xi = X$ for all $\xi \in \alpha \setminus \{\xi_0, \xi_1, \ldots, \xi_k\}$. Define

$$\mathcal{W} = \bigcap_{i=0}^{k} (U_{\xi_i} \cap O_{\xi_i})^-.$$

Then $\mathcal{W}$ is a τ_V-open neighborhood of F. Since F is an accumulation point of $(\bar{x}_n\colon n < \omega)$, for any $n < \omega$, there is a $j_n \geqslant n$ such that $\bar{x}_{j_n} \in \mathcal{W}$. Consequently, we have

$$\bigl\{ x_{j_n}(\xi)\colon \xi < \alpha \bigr\} \cap (U_{\xi_i} \cap O_{\xi_i}) \neq \emptyset$$

for all $i \leqslant k$. It follows that $x_{j_n}(\xi_i) \in O_{\xi_i}$ for all $i \leqslant k$. Therefore, $x_{j_n} \in O$, and thus x is an accumulation point of $(x_n\colon n < \omega)$ in X^α. However, this is a contradiction, since $(x_n\colon n < \omega)$ is a closed discrete sequence in X^α. $\square$

Proof of Theorem 2.1. Suppose that $\prod\{X_i\colon i \in A\}$ is not countably compact. Then there exists a closed discrete sequence $(x_n\colon n < \omega)$ in $X^{|A|}$, where $x_n = \langle x_n(i) \rangle_{i \in A}$ for each $n < \omega$, such that $x_n(i) \in X_i$ for all $n < \omega$ and all $i \in A$. Note that each X_i is clopen in X. If we take $U_i = V_i = X_i$ for all $i \in A$, and apply Lemma 2.2, then we can conclude that 2^X is not countably compact. $\square$.

Let $\omega^* = \beta\omega \setminus \omega$ be the set of free ultrafilters on ω with the relative topology of $\beta\omega$, and $\mathcal{D} \in \omega^*$. We say a point $x \in X$ a *$\mathcal{D}$-limit* of a sequence $(A_n\colon n < \omega)$ of subsets of a space X if $\{n < \omega\colon A_n \cap U \neq \emptyset\} \in \mathcal{D}$ for each open neighborhood U of x. A space X is said to be *$\mathcal{D}$-compact* (*$\mathcal{D}$-pseudocompact*) [1,6] provided that every sequence of points (nonempty open subsets) in X has a $\mathcal{D}$-limit. Every $\mathcal{D}$-compact ($\mathcal{D}$-pseudocompact) space is countably compact (pseudocompact). In addition, two ultrafilters $x, y \in \omega^*$ are called *equivalent* if there exists a automorphism $h : \beta\omega \to \beta\omega$ such that $h(x) = y$. Decompose ω^* into equivalence classes called *types*. For each $x \in \omega^*$, let $T(x)$ denote the type of x.

It is well known that for any $x \in \omega^*$, $T(x)$ is a dense, but not countably compact subspace of ω^*. Furthermore, if $x \in \omega^*$ is a non-P point, then $T(x)$ is $\mathcal{D}$-pseudocompact for some $\mathcal{D} \in \omega^*$ [7, Theorem 5.5].

The next lemma is originated to [5], the current form is taken from [13].

Lemma 2.3 [5,13]. *For each $x \in \omega^*$, let $\mathcal{F}_x = \{\mathcal{D} \in \omega^*\colon x$ is the $\mathcal{D}$-limit of some one-to-one discrete sequence of $\beta\omega\}$. Then $|\mathcal{F}_x| \leqslant \mathfrak{c}$.*

In 1967, Frolik [5] constructed a Tychonoff space X whose finite powers are countably compact, but countable infinite power is not pseudocompact (thus, not countably compact). Ginsburg [6] further showed that 2^X is not pseudocompact. In our next example, we give a Tychonoff space having the following properties: all finite powers are countably compact, the countable infinite power is pseudocompact but not countably compact.

Example 2.4. *There exists a Tychonoff space X such that all finite powers of X are countably compact and X^ω is pseudocompact, but X^ω is not countably compact*. Let Z be any dense $\mathcal{D}^\sharp$-pseudocompact subspace of ω^* for some $\mathcal{D}^\sharp \in \omega^*$ such that $|Z| = \mathfrak{c}$, and such that Z is not countably compact (for example, take Z as the type of any non-P point in ω^*). By Lemma 2.3, we have $|\bigcup\{\mathcal{F}_x\colon x \in Z\}| \leqslant \mathfrak{c}$. Pick some $\mathcal{D}_0 \in \omega^* \setminus \bigcup\{\mathcal{F}_x\colon x \in Z\}$. Let $Y_0 = \{x \in \omega^*\colon x$ is the $\mathcal{D}_0$-limit of some discrete sequence in $Z\}$. Then $Z \cap Y_0 = \emptyset$. Inductively, one can construct an ω_1-sequence $(Y_\alpha\colon \alpha < \omega_1)$ of pairwise disjoint subsets of ω^* and an ω_1-sequence $(\mathcal{D}_\alpha\colon \alpha < \omega_1)$ in ω^* with distinct types such that for every $\alpha < \omega_1$, every discrete sequence of points in $Z \cup \bigcup\{Y_\beta\colon \beta < \alpha\}$ has a $\mathcal{D}_\alpha$-limit in Y_α.

Select a sequence $(A_n\colon n < \omega)$ of subsets of ω_1 such that the intersection of any finite subfamily is unbounded, and $\bigcap_{n<\omega} A_n = \emptyset$. For each $n < \omega$, let $X_n = Z \cup \bigcup\{Y_\alpha\colon \alpha \in A_n\}$ be the subspace of $\beta\omega$. Similarly to Theorem D in [5], any finite subproduct of $\prod_{n<\omega} X_n$ is countably compact. The diagonal of $\prod_{n<\omega} X_n$ is homeomorphic to Z which is not countably compact, it follows that $\prod_{n<\omega} X_n$ is not countably compact. Since $\prod_{n<\omega} Z$ is a dense $\mathcal{D}^\sharp$-pseudocompact subspace of $\prod_{n<\omega} X_n$, then $\prod_{n<\omega} X_n$ itself is also $\mathcal{D}^\sharp$-pseudocompact.

Let $X = \bigoplus\{X_n\colon n < \omega\} \cup \{\infty\}$ be the one point countably compactification of the disjoint union $\bigoplus\{X_n\colon n < \omega\}$. It is easily checked that any finite power of X is countably compact, but the countable infinite power X^ω is not countably compact. To show that X^ω is pseudocompact, let $(G_n\colon n < \omega)$ be an arbitrary sequence of basic open sets in X^ω. We shall show that $(G_n\colon n < \omega)$ has a cluster point in X^ω. Without loss of generality, we may assume that $G_n = \prod_{i=1}^{k_n} U_{ni} \times \prod_{i \notin \{1,\ldots,k_n\}} X$ for each $n < \omega$, and $(k_n\colon n < \omega)$ is an increasing sequence of $\mathbb{N}$. Inductively, after taking refinements, one can define a sequence $(B_j\colon j < \omega)$ in $[\omega]^\omega$ and a sequence $(\varphi_j\colon j < \omega)$ of mappings such that for each $j < \omega$,

(i) $\varphi_j : B_j \to \omega$;
(ii) $|B_{j+1} \setminus B_j| < \omega$; and either
(iii) φ_j is an injective mapping such that $U_{ij} \subseteq X_{\varphi_j(i)}$ for all $i \in B_j$; or
(iv) φ_j is an constant mapping such that $U_{ij} \subseteq X_{\varphi_j(i)}$ for all $i \in B_j$.

By using the diagonal argument, one can choose some $B = \{b_n\colon n < \omega\} \in [\omega]^\omega$ such that $|B \setminus B_j| < \omega$ for all $j < \omega$. Note that each X_n is $\mathcal{D}^\sharp$-pseudocompact. Define a point $x = \langle x_j \rangle_{j<\omega} \in X^\omega$ as follows: if φ_j is an constant mapping, let x_j be the $\mathcal{D}^\sharp$-limit of $(U_{b_n j}\colon n < \omega)$ in $X_{\varphi_j(i)}$, where $i \in B_j$; and let $x_j = \infty$ if φ_j is an injective mapping. For any basic open neighborhood $V = \prod_{i=1}^m V_i \times \prod_{i>m} X$ of x in X^ω, choose some $n < \omega$ such that $k_n > m$. Since $\{n\colon U_{b_n i} \cap V_i \neq \emptyset\} \in \mathcal{D}^\sharp$ for each $i \leqslant m$, then $\{b_n \in B\colon G_{b_n} \cap V \neq \emptyset\}$ is infinite. This shows that x is a cluster point of $(G_n\colon n < \omega)$ in X^ω. Therefore, X^ω is pseudocompact.

Remark 2.5. The referee points out that the space X in Example 2.4 is in fact $\mathcal{D}^\sharp$-pseudocompact, and thus any power X^α of X is $\mathcal{D}^\sharp$-pseudocompact. To see this, let $L = \bigoplus\{Z_n\colon n \in \omega\}$ be the pairwise disjoint union of Z_n's, where $Z_n = Z$ for every $n \in \omega$ and put $H = L \cup \{\infty\}$. Equip H with the induced topology from X. Then H is a dense $D^\sharp$-pseudocompact subspace of X. This implies that X itself is $D^\sharp$-pseudocompact.

In the next example, we apply a known space in [15] and Theorem 2.1 to give a negative answer to Question 1.1 for countable compactness in one direction, even when X^{ω_1} is countably compact.

Example 2.6. *There exists a countably compact Tychonoff space X such that $X^{\mathfrak{t}}$ is countably compact but 2^X is not countably compact.* For each $\mathcal{D} \in \omega^*$, let $X_{\mathcal{D}} = \beta\omega \setminus \{\mathcal{D}\}$ be endowed with the relative topology of $\beta\omega$. Let X be the one-point countably compactification of the disjoint union $\bigoplus\{X_{\mathcal{D}}\colon \mathcal{D} \in \omega^*\}$. It is shown in [15, Example 4.13] that X is totally countably compact (A space X is *totally countably compact* if every sequence of points in X has a subsequence which is contained in a compact subset of X), but not $\mathcal{D}$-compact for any $\mathcal{D} \in \omega^*$. By [15, Theorem 3.3], $X^{\mathfrak{t}}$ is countably compact (Recall that $\mathfrak{t}$ is the minimal cardinality of towers and $\mathfrak{t} \geqslant \omega_1$). Let $Z = \prod\{X_{\mathcal{D}}\colon \mathcal{D} \in \omega^*\}$. As shown in [15, Example 3.14], Z is not countably compact. By Theorem 2.1, we conclude that 2^X is not countably compact.

The following lemma is well known.

Lemma 2.7 [7]. *A space X is $\mathcal{D}$-compact for some $\mathcal{D} \in \omega^*$ if and only if $X^{2^{\mathfrak{c}}}$ is countably compact.*

It is shown in [6] that for any space X, the hyperspace 2^X is $\mathcal{D}$-compact ($\mathcal{D}$-pseudocompact) if and only if X is $\mathcal{D}$-compact ($\mathcal{D}$-pseudocompact). It follows immediately from Lemma 2.7 that 2^X is countably compact provided that $X^{2^{\mathfrak{c}}}$ is countably compact. Now, a natural question which arises from this fact is:

Question 2.8. *Is $2^{\mathfrak{c}}$ the best possible cardinal for the power of a countably compact space X to guarantee the countable compactness of 2^X?*

The answer to Question 2.8 is "yes" by Example 2.6, if we assume the first two steps of **GCH** (i.e., $\mathfrak{c} = 2^\omega = \omega_1$, and $2^{\omega_1} = \omega_2$). To get a better consistency answer, we need an extra concept. Recall that an ultrafilter $\mathcal{D} \in \omega^*$ is said to be a *selective ultrafilter*, or a *Ramsey ultrafilter* [3] if whenever $(A_n\colon n < \omega)$ is a countable decomposition of ω and $A_n \notin \mathcal{D}$ for all $n < \omega$, there is $A \in \mathcal{D}$ with $|A \cap A_n| \leqslant 1$ for all $n < \omega$.

Example 2.9. *Assuming* **MA**, *there exists a Tychonoff space X such that for every $\alpha < 2^{\mathfrak{c}}$, X^α is countably compact but 2^X is not countably compact*. Assuming **MA**, Saks [13, Theorem 2.5] showed that there exists a $2^{\mathfrak{c}}$-sequence $(X_\varsigma\colon \varsigma < 2^{\mathfrak{c}})$ of subspaces of $\beta\omega$ such that for every proper $I \subset 2^{\mathfrak{c}}$, there is some selective ultrafilter $\mathcal{D} \in \omega^*$ ($\mathcal{D}$ depends on I) such that the partial product $\prod\{X_\varsigma\colon \varsigma \in I\}$ is $\mathcal{D}$-compact, but the full product $\prod\{X_\varsigma\colon \varsigma < 2^{\mathfrak{c}}\}$ is not countably compact. Let X be the one-point countably compactification of $\bigoplus\{X_\varsigma\colon \varsigma < 2^{\mathfrak{c}}\}$.

Since $\prod\{X_\varsigma\colon \varsigma < 2^{\mathfrak{c}}\}$ is not countably compact, by Theorem 2.1, 2^X is not countably compact. Hence, we are left to show that for any cardinal $\alpha < 2^{\mathfrak{c}}$, the αth power X^α of X is countably compact. To this end, fix an $\alpha < 2^{\mathfrak{c}}$, and pick up an arbitrary sequence $(x_n\colon n < \omega)$ in X^α, where $x_n = \langle x_n(\xi)\rangle_{\xi<\alpha}$ for all $n < \omega$. We shall deduce that $(x_n\colon n < \omega)$ has an accumulation point in X^α. Note that a sequence in a product space has a $\mathcal{D}$-limit for some $\mathcal{D} \in \omega^*$ if and only if its projection on each factor has a $\mathcal{D}$-limit (with respect to the same $\mathcal{D}$). Thus, it suffices to prove that there is some $\mathcal{D} \in \omega^*$ such that for any $\xi < \alpha$, $(x_n(\xi)\colon n < \omega)$ has a $\mathcal{D}$-limit in X. To do so, we first choose a proper subset $I \subset 2^{\mathfrak{c}}$ such that

$$\{x_n(\xi)\colon \xi < \alpha,\ n < \omega\} \subseteq \bigcup\{X_\zeta\colon \zeta \in I\} \cup \{\infty\}.$$

Let $\mathcal{D} \in \omega^*$ be a selective ultrafilter such that $\prod\{X_\zeta\colon \zeta \in I\}$ is $\mathcal{D}$-compact. For a fixed $\xi < \alpha$, define $A_\xi = \{n < \omega\colon x_n(\xi) \neq \infty\}$. Since $\mathcal{D}$ is an ultrafilter, we have

(i) $\omega \setminus A_\xi \in \mathcal{D}$, or
(ii) $A_\xi \in \mathcal{D}$.

If (i) holds, then ∞ is the $\mathcal{D}$-limit of $(x_n(\xi)\colon n < \omega)$ in X. If (ii) holds, since we assume that all X_ς's are pairwise disjoint in $\bigoplus\{X_\varsigma\colon \varsigma < 2^{\mathfrak{c}}\}$, we first define a countable subset

$$I(\xi) = \{\varsigma \in I\colon \{x_n(\xi)\colon n \in A_\xi\} \cap X_\varsigma \neq \emptyset\}$$

of I and then decompose ω into a countable disjoint union as

$$\omega = (\omega \setminus A_\xi) \cup \bigcup\{\{n \in A_\xi\colon x_n(\xi) \in X_\varsigma\}\colon \varsigma \in I(\xi)\}.$$

Now, we have the following two subcases:

(ii$_a$) There exists some $\varsigma \in I(\xi)$ with $\{n \in A_\xi\colon x_n(\xi) \in X_\varsigma\} \in \mathcal{D}$. In this case, X_ς is $\mathcal{D}$-compact, thus $(x_n(\xi)\colon n < \omega)$ has a $\mathcal{D}$-limit in $X_\varsigma \subset X$.
(ii$_b$) For every $\varsigma \in I(\xi)$, $\{n \in A_\xi\colon x_n(\xi) \in X_\varsigma\} \notin \mathcal{D}$. In this case, $|I(\xi)| = \omega$. Since $\mathcal{D}$ is a selective ultrafilter, there exists some $B_\xi \in \mathcal{D}$ and an injective mapping $f : B_\xi \to I(\xi)$ with $x_n(\xi) \in X_{f(n)}$ for all $n \in B_\xi$. It is easy to see that ∞ is the $\mathcal{D}$-limit of $(x_n(\xi)\colon n < \omega)$ in X.

It follows that in any case, $(x_n\colon n < \omega)$ has a $\mathcal{D}$-limit in X^α, which implies that it has an accumulation point in X^α. Therefore, X^α is countably compact.

Remark 2.10. It remains an open question whether the answer to Question 2.8 is affirmative in **ZFC**. Moreover, it is not clear to the authors that whether the hyperspace of the space given in Example 2.4 or Example 2.6 is pseudocompact or not. In fact, it is still an unsolved problem that whether 2^X is pseudocompact for a Tychonoff space X whenever all powers of X are pseudocompact.

3. Positive results for homogeneous spaces

In this section, we shall provide some positive answers to Question 1.1 in some special cases. Given a space X, let id_X be the identity mapping on X. The family of all homeomorphisms of X onto itself will be denoted by $\mathrm{Aut}(X)$. Recall that X is said to be *homogeneous* if for any two distinct points $x, y \in X$ there exists an $f \in \mathrm{Aut}(X)$ such that $f(x) = y$.

Theorem 3.1. *Let X be a regular homogeneous space. If 2^X is countably compact, then X^ω is countably compact.*

Proof. If X is finite, we have nothing to prove. So, we assume that X is infinite. Then, by the homogeneity of X and the countable compactness of 2^X, X must have no isolated points. By [6, Corollary 2.3], X is countably compact.

To prove that X^ω is countably compact, let $(x_n\colon n < \omega)$ be an arbitrary sequence in X^ω, where $x_n = \langle x_n(k)\rangle_{k<\omega}$ for each $n < \omega$. We shall deduce that $(x_n\colon n < \omega)$ has an accumulation point in X^ω. Let $z(0)$ be an accumulation point of the sequence $(x_n(0)\colon n < \omega)$. Then choose two open subsets $V_0, U_0 \subseteq X$ such that $z(0) \in V_0 \subseteq \overline{V}_0 \subseteq U_0$ and $\overline{U}_0 \neq X$. Next, we define

$$A_0 = \bigl\{n < \omega\colon f_0\bigl(x_n(0)\bigr) \in V_0\bigr\},$$

where $f_0 = \mathrm{id}_X$ (for the sake of unification). Let $z(1)$ be an accumulation point of the sequence $(x_n(1)\colon n \in A_0)$. Select an $f_1 \in \mathrm{Aut}(X)$ and two open subsets $V_1, U_1 \subseteq X$ such that $f_1(z(1)) \in V_1 \subseteq \overline{V}_1 \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and $\overline{U}_0 \cup \overline{U}_1 \neq X$. (In case $z(1) \notin \overline{U}_0$, then just take $f_1 = \mathrm{id}_X$.) Then we define

$$A_1 = \bigl\{n \in A_0\colon f_1\bigl(x_n(1)\bigr) \in V_1\bigr\}.$$

Since X has no isolated points, we can continue this process inductively, and thus obtain infinite sequences $(z(j)\colon j < \omega)$, $(U_j\colon j < \omega)$, $(V_j\colon j < \omega)$, $(A_j\colon j < \omega)$ and $(f_j\colon j < \omega)$ such that

(i) $f_j \in \mathrm{Aut}(X)$ for all $j < \omega$,
(ii) $(U_j\colon j < \omega)$ is a sequence of pairwise disjoint nonempty open sets in X with $\bigcup_{j\leqslant n} \overline{U}_j \neq X$ for all $n < \omega$,
(iii) $(V_j\colon j < \omega)$ is a sequence of nonempty open sets in X with $\overline{V}_j \subseteq U_j$ for all $j < \omega$,

(iv) for each $j < \omega$, A_j is an infinite subset of ω and $A_{j+1} \subseteq A_j$,
(v) $A_{j+1} = \{n \in A_j\colon f_{j+1}(x_n(j+1)) \in V_{j+1}\}$ for all $j < \omega$,
(vi) for each $j < \omega$, $z(j)$ is an accumulation point of $(x_n(j)\colon n \in A_j)$, and
(vii) $f_j(z(j)) \in U_j$ for all $j < \omega$.

Now, one can pick an infinite set $A = \{a_j\colon j < \omega\} \subseteq \omega$ such that $a_j \in A_j$ for all $j < \omega$. Then, by (iv), $|A \setminus A_j| < \omega$ for all $j < \omega$. For each $n < \omega$ and $j < \omega$, let $y_{a_n}(j) = f_j(x_{a_n}(j))$ and $y_{a_n} = \langle y_{a_n}(j)\rangle_{j<\omega}$.

Next, we shall show that the sequence $(y_{a_n}\colon n < \omega)$ has an accumulation point in X^ω by using Lemma 2.2. To this end, we choose an infinite sequence $(t_{a_n}\colon n < \omega)$ in X^ω such that for each $j < \omega$, $|\{n \in \omega\colon t_{a_n}(j) \neq y_{a_n}(j)\}| < \omega$, and $t_{a_n}(j) \in V_j$ for all $n, j < \omega$. This is possible, since for each $j < \omega$, V_j is infinite, and $y_{a_n}(j) \in V_j$ for all but finitely many $n < \omega$. It is easy to see that $(t_{a_n}\colon n < \omega)$ has an accumulation point in X^ω if and only if $(y_{a_n}\colon n < \omega)$ does. Since 2^X is countably compact, the sequence $(t_{a_n}\colon n < \omega)$ has an accumulation point in X^ω, which is also an accumulation point of $(y_{a_n}\colon n < \omega)$.

Let $y = \langle y(j)\rangle_{j<\omega}$ be any accumulation point of $(y_{a_n}\colon n < \omega)$ in X^ω. It can be easily checked that $x = \langle f_j^{-1}(y(j))\rangle_{j<\omega}$ is an accumulation point of $(x_{a_n}\colon n < \omega)$ in X^ω, and thus is also an accumulation point of $(x_n\colon n < \omega)$ in X^ω. Therefore, X^ω is countably compact. □

By replacing the closed discrete sequence of points in Lemma 2.2 with a suitable sequence of open sets, we obtain the following lemma.

Lemma 3.2. *Let X be a space without isolated points. For all $j, n < \omega$, let V_j, U_j and O_{nj} be nonempty open subsets of X satisfying*

(i) *$(U_j\colon j < \omega)$ is pairwise disjoint,*
(ii) *$\overline{V}_j \subseteq U_j$ for all $j < \omega$, and*
(iii) *for each $j < \omega$, $O_{nj} \subseteq V_j$ for all but finitely many $n < \omega$.*

Let $(k_n\colon n < \omega)$ be a strictly increasing sequence in ω. For each $n < \omega$, define

$$O_n = \prod_{j=0}^{k_n} O_{nj} \times \prod\{X_j\colon j > k_n\},$$

where $X_j = X$ for every $j > k_n$. If 2^X is pseudocompact, then $(O_n\colon n < \omega)$ has an accumulation point in X^ω.

Proof. After making some adjustment to the matrix (O_{nj}) of open sets in a manner similar to what we have done in Theorem 3.1, we may require that $O_{nj} \subseteq V_j$ for all $n, j < \omega$. Let $\mathcal{G}_n = \{O_{n0}, \ldots, O_{nk_n}\}$ for each $n < \omega$. Since 2^X is pseudocompact, $(\langle\mathcal{G}_n\rangle\colon n < \omega)$ has an accumulation point $F \in 2^X$. We claim that $F \cap \overline{V}_j \neq \emptyset$ for all $j < \omega$. Suppose the contrary. Then $F \cap \overline{V}_{j_0} = \emptyset$ for some $j_0 < \omega$. It follows that $F \in (X \setminus \overline{V}_{j_0})^+$. Hence, we obtain an infinite subset

$$I = \{n > j_0\colon \langle\mathcal{G}_n\rangle \cap (X \setminus \overline{V}_{j_0})^+ \neq \emptyset\}$$

of ω. Pick any $n \in I$ and an arbitrary $F_n \in \langle \mathcal{G}_n \rangle \cap (X \setminus \overline{V}_{j_0})^+$. Then, we have $\emptyset \neq F_n \cap O_{nj_0} \subseteq V_{j_0}$. But this is impossible, since $F_n \in (X \setminus \overline{V}_{j_0})^+$.

Next, select a point $x(j) \in F \cap \overline{V}_j$ for each $j < \omega$, and let $x = \langle x(j) \rangle_{j<\omega}$. We show that x is an accumulation point of $(O_n\colon n < \omega)$ in X^ω. To do this, let

$$W = \prod_{j=0}^{k} W_j \times \prod\{X_j\colon j > k\}$$

be an arbitrary basic open neighborhood of x in X^ω, where W_j ($j \leqslant k$) is an open subset in X with $x(j) \in W_j$ and $X_j = X$ for all $j > k$. Let $\mathcal{W} = \bigcap_{j=0}^{k} (U_j \cap W_j)^-$. Then $\mathcal{W}$ is a τ_V-open neighborhood of F. Since F is an accumulation point of the sequence $(\langle \mathcal{G}_n \rangle\colon n < \omega)$, we have an infinite subset $J = \{n > k\colon \langle \mathcal{G}_n \rangle \cap \mathcal{W} \neq \emptyset\}$ of ω. For any $n \in J$, we pick up an arbitrary $H_n \in \langle \mathcal{G}_n \rangle \cap \mathcal{W}$, and decompose H_n into a finite disjoint union as $H_n = \bigcup_{j=0}^{k_n} (H_n \cap O_{nj})$. Since

$$(U_j \cap W_j) \cap \Big(\bigcup \{H_n \cap O_{ni}\colon i \neq j,\ 0 \leqslant i \leqslant k_n\} \Big) = \emptyset$$

and $(U_j \cap W_j) \cap H_n \neq \emptyset$, then $(H_n \cap O_{nj}) \cap (U_j \cap W_j) \neq \emptyset$ for every $j \leqslant k$. Thus $O_n \cap W \neq \emptyset$ for all $n \in J$, and x is an accumulation point of $(O_n\colon n < \omega)$. □

The proof our next theorem is similar to that of Theorem 3.1.

Theorem 3.3. *Let X be a Tychonoff homogeneous space. If 2^X is pseudocompact, then X^α is pseudocompact for any cardinal α.*

Proof. As we have done in Theorem 3.1, to avoid triviality, we assume that X is infinite, and thus it has no isolated points. By [6, Corollary 2.7], all finite powers of X are pseudocompact. It suffices to show that X^ω is pseudocompact, since an infinite power X^α of a Tychonoff space X is pseudocompact if and only if X^ω is pseudocompact. To do so, let $(G_n\colon n < \omega)$ be a sequence of nonempty basic open sets of X^ω. For each $n < \omega$, we may assume that

$$G_n = \prod_{i=0}^{k_n} G_{ni} \times \prod\{X_i\colon i > k_n\},$$

where each $k_n < \omega$, each G_{ni} ($i \leqslant k_n$) is a nonempty open subset of X and $X_i = X$ for every $i > k_n$. Without loss of generality, we may require that $(k_n\colon n < \omega)$ is strictly increasing. (In fact, for each $n < \omega$, we can always add some more open sets G_{ni} with $G_{ni} = X$ if necessary.) Let $z(0)$ be an accumulation point of the sequence $(G_{n0}\colon n < \omega)$. By the regularity of X, we can choose two open sets V_0, U_0 of X such that $z(0) \in V_0 \subseteq \overline{V}_0 \subseteq U_0$, and $\overline{U}_0 \neq X$. Then, we define

$$A_0 = \big\{n < \omega\colon f_0(G_{n0}) \cap V_0 \neq \emptyset\big\},$$

where $f_0 = \mathrm{id}_X$. For each $n \in A_0$, let $O_{n0} = f_0(G_{n0}) \cap V_0$. Let $z(1)$ be an accumulation point of the sequence $(G_{n1}\colon n \in A_0)$. Select an $f_1 \in \mathrm{Aut}(X)$ and two open subsets V_1, U_1

of X such that $f_1(z(1)) \in V_1 \subseteq \overline{V}_1 \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and $\overline{U}_0 \cup \overline{U}_1 \neq X$. (In case that $z(1) \notin \overline{U}_0$, then just take $f_1 = \mathrm{id}_X$.) Let

$$A_1 = \{n \in A_0\colon\ f_1(G_{n1}) \cap V_1 \neq \emptyset\}.$$

For each $n \in A_1$, let $O_{n1} = f_1(G_{n1}) \cap V_1$. Since X has no isolated points, one can continue this process inductively, which yields infinite sequences $(z(j)\colon\ j < \omega)$, $(V_j\colon\ j < \omega)$, $(U_j\colon\ j < \omega)$, $(A_j\colon\ j < \omega)$, $(f_j\colon\ j < \omega)$ and $(O_{nj}\colon\ n \in A_j)$ (for each $j < \omega$) such that

(i) $f_j \in \mathrm{Aut}(X)$ for all $j < \omega$,
(ii) $(U_j\colon\ j < \omega)$ is a sequence of pairwise disjoint nonempty open sets in X with $\bigcup_{j \leqslant n} \overline{U}_j \neq X$ for all $n < \omega$,
(iii) $(V_j\colon\ j < \omega)$ is a sequence of nonempty open sets in X with $\overline{V}_j \subseteq U_j$ for all $j < \omega$,
(iv) for each $j < \omega$, A_j is an infinite subset of ω and $A_{j+1} \subseteq A_j$,
(v) $A_{j+1} = \{n \in A_j\colon\ f_{j+1}(G_{n(j+1)}) \cap V_{j+1} \neq \emptyset\}$ for all $j < \omega$,
(vi) for each $j < \omega$, $z(j)$ is an accumulation point of $(G_{nj}\colon\ n \in A_j)$,
(vii) $f_j(z(j)) \in V_j$ for all $j < \omega$, and
(viii) for each $j < \omega$, $O_{nj} = f_j(G_{nj}) \cap V_j$ for all $n \in A_j$.

Now, we can pick an infinite set $A = \{a_j\colon\ j < \omega\} \subseteq \omega$ such that $a_j \in A_j$ for every $j < \omega$. For each $n < \omega$, let $O_{a_n} \subseteq X^\omega$ be defined by

$$O_{a_n} = \prod_{j=0}^{n} O_{a_n j} \times \prod_{j=n+1}^{k_{a_n}} f_j(G_{a_n j}) \times \prod\{X_j\colon\ j > k_{a_n}\},$$

where $X_j = X$ for every $j > k_{a_n}$. Moreover, define $Q_{a_n} \subseteq X^\omega$ as

$$Q_{a_n} = \prod_{j=0}^{k_{a_n}} f_j(G_{nj}) \times \prod\{X_j\colon\ j > k_{a_n}\}$$

for each $n < \omega$, where $X_j = X$ for all $j > k_{a_n}$.

It is clear that $(U_j\colon\ j < \omega)$, $(V_j\colon\ j < \omega)$, $(O_{a_n j}\colon\ n, j < \omega)$ and $(O_{a_n}\colon\ n < \omega)$ satisfy conditions in Lemma 3.2. By the pseudocompactness of 2^X, $(O_{a_n}\colon\ n < \omega)$ has an accumulation point in X^ω, say $y = \langle y(j)\rangle_{j<\omega}$. Since $O_{a_n} \subseteq Q_{a_n}$ for all $n < \omega$, y is also an accumulation point of $(Q_{a_n}\colon\ n < \omega)$. It follows that the point $x = \langle f_j^{-1}(y(j))\rangle_{j<\omega}$ is an accumulation point of $(G_{a_n}\colon\ n < \omega)$ in X^ω, thus is an accumulation point of $(G_n\colon\ n < \omega)$. Therefore, X^ω is pseudocompact. □

In contrast to counterexamples in Section 2, Theorem 3.1 and Theorem 3.3 give positive answers to Question 1.1 for countable compactness and pseudocompactness respectively in the class of homogeneous Tychonoff spaces in the other direction.

Remark 3.4. Recall that a space X is $\mathcal{G}$-*pseudocompact* [6] if every locally finite family of nonempty open sets is finite. Every $\mathcal{G}$-pseudocompact space is pseudocompact. For Tychonoff spaces, these two notions are equivalent. In the literature, $\mathcal{G}$-pseudocompact spaces are also called *feebly compact*. When X is a Tychonoff space, 2^X is pseudocompact if and only if 2^X is $\mathcal{G}$-pseudocompact [6, Proposition 2.6].

Theorem 2.5 of [6] claims that if X is regular and 2^X is $\mathcal{G}$-pseudocompact, then all finite powers of X are $\mathcal{G}$-pseudocompact. By applying an argument similar to that in Lemma 3.2, one can give an alternative proof to this theorem. In fact, Theorem 3.3 is still valid when X is regular and pseudocompactness is replaced by $\mathcal{G}$-pseudocompactness.

4. Additional remarks

In [6], Ginsburg also posed the following interesting question.

Question 4.1. *Characterize those spaces X whose hyperspaces 2^X are countable compact (pseudocompact).*

In 1998, Natsheh considered Question 4.1 and claimed to provide a sufficient condition for 2^X to be pseudocompact (see *Questions Answers Gen. Topology* 16 (1998) 213–217 for details). According to Natsheh, a subset C in a topological space X is called a *C_δ-set* if there exists a sequence $(V_n\colon n < \omega)$ of nonempty open sets in X such that $C = \bigcap_{n<\omega} \overline{V}_n$. What he claimed to prove is the following: *If X is a pseudocompact normal space and for each sequence $(C_n\colon n < \omega)$ of C_δ-sets in X there exists an $F \in 2^X$ such that $F \cap C_n \neq \emptyset$ for all $n < \omega$ and $F \subseteq \bigcup_{n<\omega} C_n$, then 2^X is pseudocompact.*

Remark 4.2. *There is a gap in Natsheh's proof*. In fact, he used the following false statement in his argument: A Tychonoff space X is pseudocompact if and only if $\bigcap_{n<\omega} \overline{G}_n \neq \emptyset$ for every non-increasing sequence $(G_n\colon n < \omega)$ of nonempty "basic" open sets of X. Therefore, Question 4.1 remains open.

Remark 4.3. There are no analogs to Lemma 2.7 for pseudocompactness. This has been shown in [7, Example 4.4].

It is interesting that the countable compactness (pseudocompactness) of the hyperspace 2^X of a space X with respect to a topology weaker than τ_V, called the Fell topology, is completely characterized. Recall that the *Fell topology* [4] on 2^X, denoted by τ_F, is generated by taking

$$\mathcal{S} = \{U^+\colon \emptyset \neq U \subseteq X \text{ is open}, X \setminus U \text{ is compact}\} \cup \{V^-\colon \emptyset \neq V \subseteq X \text{ is open}\}$$

as a subbase. Note that τ_F is a Hausdorff topology on 2^X if and only if X is locally compact. Hou [8], and Holá and Künzi [9] showed independently that for a T_1 space X the hyperspace 2^X is countably compact with respect to τ_F if and only if X is countably compact. Moreover, Hou [8] also proved that for a locally compact space X, 2^X is pseudocompact with respect to τ_F if and only if X is either pseudocompact or not σ-compact.

References

[1] A. Bernstein, A new kind of compactness for topological spaces, Fund. Math. 66 (1970) 185–193.
[2] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic, Dordrecht, 1993.
[3] W. Comfort, S. Negrepontis, The Theory of Ultrafilters, Springer, Berlin, 1974.
[4] J. Fell, A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space, Proc. Amer. Math. Soc. 13 (1962) 472–476.
[5] Z. Frolik, Sums of ultrafilters, Bull. Amer. Math. Soc. 73 (1967) 87–91.
[6] J. Ginsburg, Some results on the countable compactness and pseudocompactness of hyperspaces, Canad. J. Math. 27 (1975) 1392–1399.
[7] J. Ginsburg, V. Saks, Some applications of ultrafilters in topology, Pacific J. Math. 57 (1975) 403–418.
[8] J. Hou, A study of hyperspaces with the Fell topology, Ph.D. Thesis, Ehime University, Japan, 1998.
[9] L'. Holá, H.P. Künzi, Properties related to compactness in hyperspaces, Topology Proc. 23 (1998) 191–205.
[10] J. Keesling, Normality and properties related to compactness in hyperspaces, Proc. Amer. Math. Soc. 24 (1970) 760–766.
[11] E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182.
[12] J. Novák, On the Cartesian product of two countably compact spaces, Fund. Math. 40 (1953) 106–112.
[13] V. Saks, Products of countably compact spaces, Topology Proc. 4 (1979) 553–575.
[14] H. Terasaka, On Cartesian products of compact spaces, Osaka Math. J. 4 (1952) 11–15.
[15] J. Vaughan, Countably compact ans sequentially compact spaces, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984.